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Prüfer Sequence and Graceful Graphs

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Abstract: Prüfer in 1918 showed one-one mapping between a labeled tree of n vertices and (n-2) tuples of labels of vertices. An unproven conjecture "The Graceful Tree Conjecture" is the most elusive conjecture in the realm of the graph labeling technique. In this paper, we have tried to correlate Prüfer encoding scheme and graceful theme of trees.

Keywords: Graph labeling, graceful labeling, graceful tree, Prüfer encoding scheme

I. INTRODUCTION

1.1 Back Ground of Prüfer Encoding Theme

Labeled trees are of greatest interest in the field of computer science, coding theory and other related areas of technology. In 1918 Prüfer (Prüfer 1918) showed injective mapping between labels of a tree with n vertices and (n-2) tuples of the tree. Prüfer sequence ancode a tree by iteratively deleting the pendant vertices of smallest label of the tree and recording the labels of their neighbors until only one edge remains. Naville (Neville 1953) proposed three different methods for encoding a given tree in which first method is equivalent to Prüfer encoding. Deo and Micikevicius proposed a new encoding technique of a tree by using a stack and queue. One of the most important drawback of Prüfer encoding scheme is to find the center and diameter of a given labeled tree with given Prüfer code. Prüfer codes are also used to generate random trees and random connected graph (Kumar et al 1998). A number of properties of trees such as radius, diameter, leaves and degree distribution, etc. can be easily determined for tree codes (Moon

1970). It is known that there are n^{n-2} labeled trees with a fixed set of n vertices. This is the Cayley's Tree formula (Cayley 1889). It is noted that Prüfer sequence (Prüfer 1918) is a sequence of n-2 terms for a given n vertex tree and each being one of the number from 1 to n. So, there are n^{n-2} Prüfer sequence for a given n.

1.1.1. An algorithm for finding preüfer sequence (encoding) from a labeled tree

1. Take any tree T_n whose vertices are labeled from 1 to n, where n ∈ N.

2. Take a vertex of degree one with smallest label, delete it from the tree and note down its adjacent vertex.

3. Repeat the steps 1 and 2 until any one edge remains or two vertices remains.

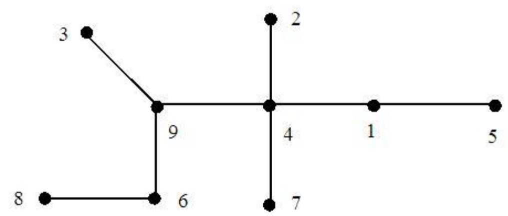

Fig. 1 Labeled tree with 9 vertices

Thus, it is noted that there are n-2 terms in the Prüfer sequence of a tree with n vertices and some vertex appear more than once. The degree of those vertices which appear more than once is one plus the number of times they appear in the sequence. It is also noticed that all the values in the Prüfer sequence are the labels of the internal vertices. If T_n be a tree with n vertices, then from Cayley's true formula, the number of distinct labeled tree is n^{n-2}.

1.1.2. An algorithm for constructing a labeled tree from Prüfer sequence (encoding)

We can also generate a labeled tree from a given Prüfer encoding (sequence). The algorithm has the following steps :

1. Analyze the least label from 1 to n that does not appear in the Prüfer sequence P.
2. Attach this vertex to the first label in P.
3. Remove the first label of P.
4. Repeat this process until there are no labels left in P.
5. Attach the last number in P to the vertex which has the label n.

1.1.3. Prüfer like encoding for a tree

Naville (Naville 1953) gave three different methods for encoding a labeled tree. In all of the three methods, an arbitrary vertex is chosen as a root and this root is never deleted from the tree. When the root vertex is chosen in such a way that it gets the largest label, then Naville's first method is equivalent to Prüfer

method. One of the drawback of Prüfer encoding scheme is the lack of determining the diameter or center of a tree only when the code is given (without drawing a tree from the code). In Naville's second encoding scheme this lack is removed, i.e. one can easily find the diameter or the center of a tree if the code is given. Naville's second method has r stages where r is the radius of the tree. Remove all the leaves in ascending order in each stage of a labeled tree and record the label of their adjacent vertices. This process is repeated until only one edge remains. Third encoding scheme due to Naville (Naville 1953) starts by deleting the leaf with the smallest label among all the leaves with the given labels and record the label of its neighbor w. If w is again a leaf of the resulting subtree, then delete w, otherwise leaf with smallest label is deleted from the resulting tree. For example, consider a tree with 9 vertices and these vertices are labeled from 1 to 9 as shown in Fig 1. Prüfer sequence and Naville's encoding scheme are given below :

Prüfer Encoding	4,9,1,4,4,9,6
Naville 1^{st} Encoding	4,9,1,4,4,9,6
Naville 2^{nd} Encoding	4,9,1,4,4,6,9
Naville 3^{nd} Encoding	4,9,1,4,4,9,6

1.2. Back Ground of Graceful Theme

Rosa (Rosa 1967) called a function f a β-valuation of a graph G with q edges if f is an injection from the set of vertices V(G) into the set {0,1,2,…..q} such that each edge e = uv ∈ E(G), ∀ u, v ∈ V(G) assigned the absolute difference of the label of vertices of G, then the resulting edge labels are distinct. Golomb (Golomb 1972) called such labeling graceful and this is now a popular term. In 1963 Ringel – Kotzig conjectured that all trees are graceful. This conjecture is still remains unsolved. Some known, graceful trees are: path, caterpillar (Rosa 1967), lobster (Bermond 1979) every tree of diameter at most 5 (Hrnčiar and Haviar 2001), symmetrical tree (Stanton and Zarnke 1973) etc. Kumar et al in 2010 obtained a graceful rooted tree in which every level contains same number of pendent vertices and they also obtained a larger graceful 1-distant tree from isomorphic copies $K_{1,n}$. A class of graceful rooted tree with diameter D where every vertex has even degree except for one root and the leaves in level $\left\lfloor \frac{D}{2} \right\rfloor$ has been given in (Balbuena et al 2007). Eshghi and Azimi (Eshghi and Azimi 2004, 2007) discuss a programming model and verify that all trees with 30, 35, or 40 vertices are graceful. Pradhan and Kumar (Pradhan and Kumar

2012) obtained graceful graphs from the graceful path by using series extensions and also obtained in (Pradhan and Kumar 2011) a graceful hairy cycles with pendant edges from a graceful caterpillar. An excellent survey for graceful labeling is given in (Gallian 2017).

As well, it is conjectured that all trees have graceful labeling. An n – vertex tree has (n-1)! graceful labelings [20]. By Cayley's Tree Formula, there are n^{n-2} distinct labeled trees of a n - vertex tree. If T is a tree with n vertices and P_n be its Prüfer sequence with (n-2) terms, then it is observed that a bijection exists between a tree of n vertices and its Prüfer sequence P_n. In this paper our main work is to develop a sequence analogues to Prüfer sequence of a graceful tree and find the relation between them.

RESULTS

Theorem 2.1. The least common multiple (LCM) of the terms of the generated Prüfer sequence (except 0 in Prüfer sequence) obtained from a graceful tree is divisible by the diameter of the tree.

Proof. Consider a graceful tree T with n vertices. Since vertex of the tree T has label from the set {0, 1, 2,.., (n-1)}. Applying Prüfer encoding theme in the graceful tree T. We obtain a Prüfer sequence and find the LCM of the remaining terms (except 0 label). It is noted that at least one term in the sequence has a prime factorization contains the diameter of the given graceful tree. Hence the theorem.

Theorem 2.2. If T be a graceful tree with n vertices and t be the number of distinct terms in its Prüfer sequence then the tree T has (n-t) pendent vertices either the graceful tree is given or its Prüfer code is given.

Proof. Since all the terms in Prüfer sequence of an n-vertex graceful tree are the internal vertices (the vertices with degree at least 2). Let t be the number of distinct terms in the Prüfer sequence of the graceful tree T. Hence a graceful tree T with n vertices with a given Prüfer sequence has (n-t) pendant vertices.

Conjecture 2.3. All trees with same number of pendant vertices are graceful.

Observation 2.4. Any graceful tree with a given Prüfer code has at least two pendant vertices.

Observation 2.5. We can find the total number of pendant vertices from a given Prüfer code of a graceful tree. Let t be the total number of terms in Püfer code in which m terms are distinct. So, the total number of pendant vertices are t-m+2.

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