Uttaranchal Journal of Applied and Life Sciences
U. J. App. Life. Sci. Vol: 3 (1): 66-75 **Dec 2022**

Numerical view of Lucas-Lehmer polynomials with its characteristics

Ashish Rayal[1]*, Shivam Tamta[2], Shivam Rawat[3], Mohd Kashif[4]

[1,2,3,4] Department of Mathematics, School of Applied and Life Sciences, Uttaranchal University, Dehradun-248007, Uttarakhand, India.

***Corresponding Author:** ashishrayal@uttaranchaluniversity.ac.in

Received: (20 Sept 2022) Revised: (20 Oct 2022) Accepted: (5 Dec 2022)

Abstract: This chapter presents Lucas-Lehmer polynomials and its shifted form which make a series of orthogonal polynomials. The orthogonal polynomials have a big contribution in the approximation theory. We discuss and prove various essential aspects of Lucas-Lehmer polynomials such as the orthogonality, recursive relation and Parseval's identity. The operational matrix of derivative and integral for LLP is also constructed.

Keywords: Lucas-Lehmer polynomials; Function approximations; Parseval's identity; Operational matrix.

INTRODUCTION

Orthogonal functions and polynomial series (Andrews (1992), Lehmer (1930)) have shown to be very effective and powerful for the search of numerical solutions of several problems in disparate areas, from physics to engineering, from biology to economics, and so on. Various numerical schemes based on orthogonal functions and polynomial series including wavelets have been employed to acquire approximate solutions of various problems in recent years (Rivlin (1990), Babusci, et al. (2014), Debnath (2002)). Using truncated form of orthogonal functions, these problems are transformed to solve an algebraic equation system. In recent years, orthogonal functions have become efficient tool for mathematicians, engineers and physicists to analyse and solve the real-life applications (Debnath (2002), Rayal and Verma (2020a), Rayal and Verma, S.R. (2020b)). Some of them are signal analysis, human vision, system analysis, fast algorithms, pattern recognition, numerical analysis, time-frequency analysis and optimal control. Wavelets (Rayal and Verma (2022), Rayal and Verma (2020c)) are the good localized and oscillatory functions which give the basis for several spaces. Also, wavelets are capable of improving the formulation of several mathematical models and providing more accurate solutions due to its important features like compact support, orthogonality, spectral accuracy and localization. The functions which contain discontinuities are effectively approximated by wavelets basis. In the past two decades, Wavelet approaches have been extensively employed to solve integral and differential equations of integer or non-integer order arising in several engineering and scientific problems.

In this chapter, we will analyse the appearance of Lucas-Lehmer polynomials and discuss several essential characteristics such as orthogonality, recursive relation and Parseval's identity.

LUCAS LEHMER POLYNOMIALS

The Lucas-Lehmer polynomials $L_n(s)$ are defined on the interval $[-2,2]$ (Lehmer (1930)) as

$$L_n(s) = L_n^+(s) + L_n^-(s),$$

where

$$L_n^+(s) = \left(\frac{s^2}{2} - 1 + \sqrt{\left(\frac{s^2}{2} - 1\right)^2 - 1} \right)^{2^{n-1}},$$

and

$$L_n^-(s) = \left(\frac{s^2}{2} - 1 - \sqrt{\left(\frac{s^2}{2} - 1\right)^2 - 1} \right)^{2^{n-1}}.$$

Also, the modulus of both part is unit, that is,

$$\left|L_n^+(s)\right| = \left|L_n^-(s)\right| = 1.$$

WEIGHTED FUNCTION FOR LUCAS LEHMER POLYNOMIALS

The weighted function $w(s)$ for the Lucas-Lehmer polynomial (LLP) $L_n(s)$ is given by

$$w(s) = \frac{1}{4\sqrt{4 - s^2}}.$$

The LLP $L_n(s)$ are orthogonal under the weighted function $w(s)$ for $-2 \le s \le 2$.

ORTHOGONALITY OF LUCAS LEHMER POLYNOMIALS

The LLP $L_n(s)$ of degree n are complete orthogonal with a weighted function $w(s)$ for $-2 \le s \le 2$ as

$$\int_{-2}^{2} w(s)L_n(s)L_m(s)ds = \begin{cases} \frac{\pi}{2}, & m = n \\ 0, & m \neq n \end{cases}.$$

Proof: The LLP is given in section as

$$L_n(s) = \left(\frac{s^2}{2} - 1 + \sqrt{\left(\frac{s^2}{2} - 1\right)^2 - 1}\right)^{2^{n-1}} + \left(\frac{s^2}{2} - 1 - \sqrt{\left(\frac{s^2}{2} - 1\right)^2 - 1}\right)^{2^{n-1}}. \quad (1)$$

Put $s = 2\cos\theta$ in Equation (1) as

$$\begin{aligned} L_n(2\cos\theta) &= \left(\frac{(2\cos\theta)^2}{2} - 1 + \sqrt{\left(\frac{(2\cos\theta)^2}{2} - 1\right)^2 - 1}\right)^{2^{n-1}} + \left(\frac{(2\cos\theta)^2}{2} - 1 - \sqrt{\left(\frac{(2\cos\theta)^2}{2} - 1\right)^2 - 1}\right)^{2^{n-1}} \\ &= \left(\cos 2\theta + \sqrt{(\cos 2\theta)^2 - 1}\right)^{2^{n-1}} + \left(\cos 2\theta - \sqrt{(\cos 2\theta)^2 - 1}\right)^{2^{n-1}} \\ &= \left(\cos 2\theta + i\sin 2\theta\right)^{2^{n-1}} + \left(\cos 2\theta - i\sin 2\theta\right)^{2^{n-1}} \quad (2) \\ &= \left(e^{2i\theta}\right)^{2^{n-1}} + \left(e^{-2i\theta}\right)^{2^{n-1}} \\ &= 2\cos(2^n\theta). \end{aligned}$$

Using relation of Equation (2) and $s = 2\cos\theta$, we have

$$\begin{aligned} \int_{-2}^{2} w(s)\,L_n(s)L_m(s)ds &= \int_{\pi}^{0} w(2\cos\theta)\,L_n(2\cos\theta)L_m(2\cos\theta)\,(-2\sin\theta)d\theta \\ &= \int_{0}^{\pi} w(2\cos\theta)2\cos(2^n\theta)2\cos(2^m\theta)(2\sin\theta)d\theta \\ &= \int_{0}^{\pi} \frac{1}{4\sqrt{4-(2\cos\theta)^2}} 2\cos(2^n\theta)2\cos(2^m\theta)(2\sin\theta)d\theta \\ &= \int_{0}^{\pi} \frac{1}{4(2\sin\theta)} 2\cos(2^n\theta)2\cos(2^m\theta)(2\sin\theta)d\theta \\ &= \int_{0}^{\pi} \cos(2^n\theta)\cos(2^m\theta)d\theta. \end{aligned}$$

If $m = n \in \square$, then

$$\begin{aligned} \int_{0}^{\pi} \cos(2^n\theta)\cos(2^m\theta)d\theta &= \int_{0}^{\pi} \cos^2(2^n\theta)d\theta \\ &= \int_{0}^{\pi} \frac{\left(1 + \cos 2(2^n\theta)\right)}{2} d\theta \\ &= \frac{1}{2}\left(\theta + \frac{\sin(2^{n+1}\theta)}{2^{n+1}}\right)_{0}^{\pi} \quad (3) \\ &= \frac{\pi}{2}. \end{aligned}$$

If $m \neq n \in \mathbb{N}$, then

$$\int_0^{\pi} \cos(2^n\theta)\cos(2^m\theta)d\theta = \frac{1}{2}\int_0^{\pi}\left[\cos(2^n\theta + 2^m\theta) + \cos(2^n\theta - 2^m\theta)\right]d\theta$$
$$= -\frac{1}{2}\left[\frac{\sin(2^n\theta + 2^m\theta)}{2^n + 2^m} + \frac{\sin(2^n\theta - 2^m\theta)}{2^n - 2^m}\right]_0^{\pi} \quad (4)$$
$$= 0.$$

Combine the result of Equations (3-4), we get the required result.

RELATION OF LUCAS LEHMER POLYNOMIALS WITH CHEBYSHEV POLYNOMIALS

The Chebyshev polynomials (Rivlin (1990)) is given as

$$T_n(s) = T_n^+(s) + T_n^-(s)$$
$$= \frac{\left(s + \sqrt{s^2 - 1}\right)^n}{2} + \frac{\left(s - \sqrt{s^2 - 1}\right)^n}{2}, \quad n \geq 1. \quad (5)$$

Replace s by $\left(\frac{s^2}{2} - 1\right)$ and n by 2^{n-1} in Equation (5), we get

$$T_{2^{n-1}}\left(\frac{s^2}{2} - 1\right) = \frac{\left(\frac{s^2}{2} - 1 + \sqrt{\left(\frac{s^2}{2} - 1\right)^2 - 1}\right)^{2^{n-1}} + \left(\frac{s^2}{2} - 1 - \sqrt{\left(\frac{s^2}{2} - 1\right)^2 - 1}\right)^{2^{n-1}}}{2}$$
$$= \frac{L_n(s)}{2}.$$

Therefore,

$$L_n(s) = 2T_{2^{n-1}}\left(\frac{s^2}{2} - 1\right); \quad n \geq 1.$$

RECURRENCE FORMULA FOR LUCAS LEHMER POLYNOMIALS

The two terms recursive formula is

$$L_n(s) = (L_{n-1}(s))^2 - 2; \quad L_0(s) = s, \quad n \geq 1. \quad (6)$$

First four Lucas-Lehmer polynomials are calculated by using Equation (6) as

$$L_0(s) = s$$
$$L_1(s) = s^2 - 2$$
$$L_2(s) = s^4 - 4s^2 + 2$$
$$L_3(s) = s^8 - 8s^6 + 20s^4 - 16s^2 + 2$$

DERIVATIVES OF LUCAS LEHMER POLYNOMIALS

Let $L_n(s)$ be a LLP, then we have

$$L_n(s) = L_n^+(s) + L_n^-(s), \tag{7}$$

Taking differentiation of Equation (7) under ‘s’, we get

$$\frac{d}{ds}L_n(s) = \frac{d}{ds}L_n^+(s) + \frac{d}{ds}L_n^-(s),$$

Now using the definition, we have

$$\frac{d}{ds}L_n^+(s) = \frac{d}{ds}\left(\frac{s^2}{2} - 1 + \sqrt{\left(\frac{s^2}{2}-1\right)^2 - 1}\right)^{2^{n-1}}$$

$$= 2^{n-1}\left(\frac{s^2}{2} - 1 + \sqrt{\left(\frac{s^2}{2}-1\right)^2 - 1}\right)^{2^{n-2}} \times \left(s + \frac{s\left(\frac{s^2}{2}-1\right)}{\sqrt{\left(\frac{s^2}{2}-1\right)^2 - 1}}\right)$$

$$= 2^{n-1}s\left(\frac{s^2}{2} - 1 + \sqrt{\left(\frac{s^2}{2}-1\right)^2 - 1}\right)^{2^{n-1}}\left(\left(\frac{s^2}{2}-1\right)^2 - 1\right)^{-1/2}.$$

$$\frac{d}{ds}L_n^+(s) = \frac{2^{n-1}s}{\sqrt{\left(\frac{s^2}{2}-1\right)^2 - 1}}L_n^+(s)$$

$$= \frac{2^n s}{\sqrt{\left(s^2-2\right)^2 - 4}}L_n^+(s)$$

$$= \frac{2^n s}{\sqrt{s^4 - 4s^2}}L_n^+(s)$$

$$= \frac{2^n}{\sqrt{s^2 - 4}}L_n^+(s).$$

Similarly,

$$\frac{d}{ds}L_n^-(s) = -\frac{2^n}{\sqrt{s^2 - 4}}L_n^-(s).$$

Therefore,

$$\frac{d}{ds}L_n(s) = \frac{2^n}{\sqrt{s^2 - 4}}\left[L_n^+(s) - L_n^-(s)\right].$$

In this way, we can calculate higher order derivative of LLP.

FUNCTION APPROXIMATION

Any continuous function h(s) defined over $[-2, 2]$ may be expanded in terms of LLP as

$$h(s)=\sum_{n=0}^{\infty}\varepsilon_n L_n(s)$$

At discontinuous point,

$$\frac{1}{2}\left(h^{+}(s)+h^{-}(s)\right)=\sum_{n=0}^{\infty}\varepsilon_n L_n(s),$$

where the coefficients ε_n are calculated by orthogonality property of LLP as

$$\begin{aligned}\varepsilon_n &= \frac{2}{\pi}\int_{-2}^{2} w(s)h(s)L_n(s)ds\\ &= \frac{2}{\pi}\int_{-2}^{2}\frac{1}{4\sqrt{4-s^2}}h(s)L_n(s)ds.\end{aligned}$$

PARSEVAL'S IDENTITY

The Parseval's identity for series expansion $h(s)=\sum_{n=0}^{\infty}\varepsilon_n L_n(s)$ is given by

$$\int_{-\infty}^{\infty} w(s)[h(s)]^2 ds=\frac{\pi}{2}\sum_{n=0}^{\infty}\varepsilon_n^2 .$$

Proof:

$$\begin{aligned}\int_{-\infty}^{\infty} w(s)[h(s)]^2 ds &= \int_{-\infty}^{\infty}\frac{1}{4\sqrt{4-s^2}}\left[\sum_{n=0}^{\infty}\varepsilon_n L_n(s)\right]^2 ds\\ &= \sum_{n=0}^{\infty}\varepsilon_n^2\int_{-\infty}^{\infty}\frac{1}{4\sqrt{4-s^2}}L_n^2(s)ds\\ &= \sum_{n=0}^{\infty}\varepsilon_n^2\left(\frac{\pi}{2}\right)\\ &= \frac{\pi}{2}\sum_{n=0}^{\infty}\varepsilon_n^2.\end{aligned}$$

OPERATIONAL MATRIX FOR DERIVATIVE

The derivative operational matrix $D_{n\times n}$ is obtained as

$$\frac{d}{ds}L_{n\times 1}(s) \simeq D_{n\times n}L_{n\times 1}(s),$$

where $L_{n\times 1}(s)$ is the LLP and derivative operational matrix $D_{n\times n}$ is determined by

$$D_{n\times n}=\left\langle g_{n\times 1}(s),(L_{n\times 1}(s))^{T}\right\rangle_{w(s)},$$

where

$$g_{n\times 1}(s)=\frac{d}{ds}L_{n\times 1}(s),$$

and $\langle .,.\rangle_{w(S)}$ denotes the inner product in $L^2[-2,2]$ with the weighted function $w(s)$.

- To illustrate the calculation procedure, we take $n=4$.

By using definition, we get first four Lucas-Lehmer polynomials on $(-2,2)$ as

$$L_{4\times1}(s)=\left(L_1,L_2,L_3,L_4\right)^T$$
$$=\begin{pmatrix} s \\ s^2-2 \\ s^4-4s^2+2 \\ s^8-8s^6+20s^4-16s^2+2 \end{pmatrix}.$$

Now, calculate the term $g_{4\times1}(s)$ as

$$g_{4\times1}(s)=\frac{d}{ds}L_{n\times1}(s)$$
$$=\begin{pmatrix} 1 \\ 2s \\ 4s^3-8s \\ 8s^7-48s^5+80s^3-32s \end{pmatrix}.$$

Then, the derivative operational matrix $D_{n\times n}$ is calculated as

$$D_{4\times4}=\left\langle g_{4\times1}(s),(L_{4\times1}(s))^T\right\rangle_{w(s)}$$
$$=\int_{-2}^{2}[g_{4\times1}(s)L_{1\times4}(s)]\,w(s)ds$$
$$=\begin{pmatrix} 0 & 0 & 0 & 0 \\ \pi & 0 & 0 & 0 \\ 2\pi & 0 & 0 & 0 \\ 4\pi & 0 & 0 & 0 \end{pmatrix}.$$

OPERATIONAL MATRIX FOR INTEGRAL

The integral operational matrix $P_{n\times n}$ is obtained as

$$\int_0^s L_{n\times1}(s)\,ds \simeq P_{n\times n}L_{n\times1}(s)$$

where $L_{n\times1}(s)$ is the LLP and integral operational matrix $P_{n\times n}$ is determined by

$$P_{n\times n}=\left\langle g_{n\times1}(s),(L_{n\times1}(s))^T\right\rangle_{w(s)},$$

where

$$g_{n\times1}(s)=\int_0^s L_{n\times1}(s)ds$$

and $\langle .,. \rangle_{w(S)}$ denotes the inner product in $L^2[-2,2]$ with the weight function $w(s)$.

- To illustrate the calculation procedure, we take $n=4$.

By using definition, we get first four Lucas-Lehmer polynomials on $[-2,2]$ as

$$L_{4\times 1}(s)=\left(L_1,L_2,L_3,L_4\right)^T$$

$$=\begin{pmatrix} s \\ s^2-2 \\ s^4-4s^2+2 \\ s^8-8s^6+20s^4-16s^2+2 \end{pmatrix}.$$

Now, calculate the term $g_{4\times 1}(s)$ as

$$g_{4\times 1}(s)=\int_0^s L_{4\times 1}(s)ds$$

$$=\begin{pmatrix} \frac{s^2}{2} \\ \frac{s^3}{3}-2s \\ \frac{s^5}{5}-\frac{4s^3}{3}+2s \\ \frac{s^9}{9}-\frac{8s^7}{7}+4s^5-\frac{16s^3}{3}+2s \end{pmatrix}.$$

Then, the integral operational matrix $P_{n\times n}$ is calculated as

$$P_{4\times 4}=\left\langle g_{4\times 1}(s),(L_{4\times 1}(s))^T\right\rangle_{w(s)}$$

$$=\int_{-2}^{2}[g_{4\times 1}(s)L_{1\times 4}(s)]\,w(s)ds$$

$$=\begin{pmatrix} 0 & \frac{\pi}{4} & 0 & 0 \\ -\frac{\pi}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

SHIFTED LUCAS LEHMER POLYNOMIALS

In order to apply these polynomials on $(0,1]$, we define so called shifted Lucas-Lehmer polynomial $L_n^*(s)$ by changing variable s to (4s-2) as

$$L_n^*(s)=L_n(4s-2).$$

This shifted polynomial $L_n^*(s)$ of degree n are orthogonal under the weight function $w^*(s) = w(4s-2)$ on the interval $(0,1]$ as

$$\int_0^1 w^*(s)L_n^*(s)L_m^*(s)ds = \begin{cases} \frac{\pi}{8}, & m = n \\ 0, & m \neq n \end{cases}.$$

The iterative formula is

$$L_n^*(s) = (L_{n-1}^*(s))^2 - 2; \quad L_0^*(s) = 4s - 2.$$
$$L_0(s) = 4s - 2$$
$$L_1(s) = (4s-2)^2 - 2$$
$$L_2(s) = \left((4s-2)^2 - 2\right)^2 - 2$$
$$L_3(s) = \left(\left((4s-2)^2 - 2\right)^2 - 2\right)^2 - 2$$

CONCLUSION

In this chapter, we derived some more properties related to LLP. These properties are useful in computing higher order derivative and integral of LLP. We can use the above relations whenever needed some LLP or its derivative with corresponding provided related polynomials. Since the operational matrices related to LLP contains many zeroes, so these LLP would provide an extreme and easy implementation to many numerical solutions. These polynomials may be useful for approximating polynomial functions.

REFERENCES

1. Andrews, L.C. (1992) Special Functions of Mathematics for Engineers, 2nd edition. McGraw-Hill, Inc.
2. Lehmer, D.H. (1930) An Extended Theory of Lucas Functions, Ann. Math. 31, 419-448.
3. Rivlin, T.J. (1990) Chebyshev Polynomials, 2nd Edition, John Wiley and Sons, New York.
4. Babusci, D., Dattoli, G., Palma, E.D., Sabia, E. (2014) Chebyshev polynomials and generalized complex numbers, Adv. Appl. 24, 1-10.
5. Debnath, L. (2002) Wavelets Transform and Their Applications. Birkhauser, Boston.
6. Rayal, A., Verma, S.R. (2020a) An approximate wavelets solution to the class of variational problems with fractional order. J. Appl. Math. Comput. 65, 735-769.

7. Rayal, A., Verma, S.R. (2020b) Numerical study of variational problems of moving or fixed boundary conditions by Muntz wavelets. Journal of Vibration and Control. DOI: 10.1177/1077546320974792
8. Rayal, A., Verma, S.R. (2022) Two-dimensional Gegenbauer wavelets for the numerical solution of tempered fractional model of the nonlinear Klein-Gordon equation. Applied Numerical Mathematics. 174, 191-220.
9. Rayal, A., Verma, S.R. (2020c) Numerical analysis of pantograph differential equation of the stretched type associated with fractal-fractional derivatives via fractional order Legendre wavelets. Chaos, Solitons Fractals. 139. https://doi.org/10.1016/j.chaos.2020.110076