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Modified Lucas wavelets collocation Scheme for solving Benjamina Bona Mohany Partial differential equations

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Abstract: This study presents a collocation scheme using modified Lucas wavelets to approximate solutions for important Benjamina Bona Mohany equations applied in diverse physical applications. The proposed scheme transforms the problem into an algebraic system, and comparative analysis with existing methods demonstrates its effectiveness in providing accurate solutions for Benjamina Bona Mohany partial differential equations.

Keywords: Modified Lucas wavelets; Collocation points; Benjamina Bona Mohany (BBM) equation

1. Introduction

Since the 1990s, wavelet methods have been widely used to solve various partial differential equations (PDEs) crucial for modeling physical phenomena in fields like chemical physics and fluid mechanics [1]. Analyzing approximations for both linear and nonlinear PDEs is vital for understanding complex phenomena. Wavelets, known for their effective application and accurate function representation, have gained substantial attention from researchers in different scientific and technological disciplines due to the impracticality of finding exact analytical solutions for these challenging PDEs. This work focuses on BBM PDEs, widely utilized in the scientific realm to simulate various intricate physical events. In contemporary research, numerical analysis [2] is increasingly applied to solve a variety of PDEs, including BBM equations, resulting in specified solutions.

The BBM equation is as follows:

$$a_1 w_t(z,t) + b_1 w_z(z,t) + c_1 w(z,t) w_z(z,t) - d_1 w_{zzt}(z,t) = g_1(z,t)$$

where , $a_{1,} b_1, c_1, d_1$ are known constant and $g_1(z,t)$ is real–valued continuous function on $[0,1) \times [0,1)$.

Efficient numerical techniques are vital for addressing application problems related to BBM partial differential equations (PDEs). Researchers have proposed various methods, including Lie group method [3], Finite difference method [4], Backlund transformation method [5], Adomain decomposition method [6], Integral method [7], Haar wavelet method [8], and Laguerre wavelet collocation method [9,13,14]. Notably, the wavelet method (WM) is distinguished for its power and elegance. The proposed work aims to develop a modified Lucas wavelet collocation scheme for solving BBM PDEs.

The remainders of presented paper are in following manner: Section 2 discusses brief definitions of the modified Lucas wavelets (MLWs) and section 3 discusses function approximation. Explanation of proposed modified Lucas wavelets collocation scheme (MLWsCS) for solving BBM equation is described in section 4. Test example is in Section 5. Finally, conclusion is given in last.

2. Brief definition of MLWs

MLWs delineated on four arguments: $k, \hat{n}, m, z$ is denoted by $\Psi_{n,m}(z) = \Psi(k, \hat{n} = n-1, m, z)$ and can be defined as follows on $[0, 1]$ [10-12]:

$$\Psi_{n,m}(z)=\begin{cases}2^{\frac{(k-1)}{2}}\cap_m \check{\mathcal{L}}_m(2^{k-1}z-n+1), & if\ \frac{n-1}{2^{k-1}}\le z\le\frac{n}{2^{k-1}}\ ,\\ 0, & otherwise\end{cases} \quad (1)$$

where,

$$n=1,2,\dots,2^{k-1},m=0,1,\dots,\ M-1,k\in Z^{+},\ \cap_m=\left\{\begin{array}{ll}\frac{1}{\sqrt{\pi}}, & \text{both m}=0\\ \frac{-\sqrt{2}i}{\sqrt{\pi}}, & \text{both m}=\text{odd}\\ \frac{\sqrt{2}}{\sqrt{\pi}}, & \text{both m}=\text{even}\\ 0, & \text{otherwise}\end{array}\right\}, \quad (2)$$

$\check{\mathcal{L}}_m(2^{k-1}z-n+1)$are modified Lucas polynomials (MLPs) of degree m, and z stand for normalized time which is orthonormal with regard to weight function $\wp_n(z)=\wp(2^{k-1}z-n+1)=\frac{1}{\sqrt{16z-16z^2}}$ on $[0,1]$. Moreover, these MLPs are easily calculated by the following relation:

$$\check{\mathcal{L}}_m(2^{k-1}z-n+1)\ =\frac{1}{2^m}\left[\left(i(4z-2)+\sqrt{16z-16z^2}\right)^m+\left(i(4z-2)-\sqrt{16z-16z^2}\right)^m\right]. \quad (3)$$

The introduced wavelets $\Psi_{n,m}(z)$given in equation (1) are orthonormal with regard to the weight function $\wp_n(z)=\wp(2^{k-1}z-n+1)=\frac{1}{\sqrt{16z-16z^2}}$ on $L_2[0,1]$.

i.e.,

$$\int_0^1\Psi_{n,m}(z)\Psi_{n',m'}(z)\wp_n(z)dz=\begin{cases}1, & (n,m)=(n',m')\\ 0, & (n,m)\ne(n',m')\end{cases}. \quad (4)$$

Substituting the values $k=1,n=1,\ and\ M=6,$ in equation (1) applying the relations given in equations 2 and 3, we find required MLWs.

3. Function approximation

A function $f(z)$ which is square integrable delineated on $[0,1]$ may be expressed as linear combination of MLWs series as

$$f(z)\cong\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\boldsymbol{g}_{n,m}\,\Psi_{n,m}(z), \quad (5)$$

where $\boldsymbol{g}_{n,m}=\langle f(z),\Psi_{n,m}(z)\ \rangle$ are MLWs coefficients and the $\langle .,.\rangle$ indicate the inner product in $L_2[0,1]$. Next, truncate the series described above in equation (5) as follows:

$$f(z)\cong\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\boldsymbol{g}_{n,m}\,\Psi_{n,m}(z)=\boldsymbol{g}^T\Psi(z), \quad (6)$$

Where $\boldsymbol{g}$ and $\Psi(z)$ are vector of order $2^{k-1}M\times 1$, written as

$$\mathscr{g} = \left[\mathscr{g}_{1,0}, \cdots, \mathscr{g}_{1,M-1}, \mathscr{g}_{2,0}, \cdots, \mathscr{g}_{2,M-1}, \cdots, \mathscr{g}_{2^{k-1},0}, \cdots, \mathscr{g}_{2^{k-1},M-1}\right]^T, \tag{7}$$

$$\Psi(z) = \left[\Psi_{1,0}(z), \dots, \Psi_{1,M-1}(z), \Psi_{2,0}(z), \dots, \Psi_{2,M-1}(z), \dots, \Psi_{2^{k-1},0}(z), \dots, \Psi_{2^{k-1},M-1}(z)\right]^T \tag{8}$$

Similarly, an arbitrary two-variable function $w(z,t)$ defined over $[0,1) \times [0,1)$ can be expanded in terms of MLWs basis as

$$w(z,t) \approx \Psi^T(t) P \Psi(z), \tag{9}$$

where,

$$\Psi^T(t) = \left(\Psi_{1,0}(t), \dots, \Psi_{1,M-1}(t), \Psi_{2,0}(t), \dots, \Psi_{2,M-1}(t), \dots, \Psi_{2^{k-1},0}(t), \dots, \Psi_{2^{k-1},M-1}(t)\right), \tag{10}$$

and,

$$P = \left[p_{i,j}\right]_{N\times N,} N = 2^{k-1}M. \tag{11}$$

4. Explanation of proposed modified Lucas wavelets collocation scheme

In section 4, MLWsCS is used to solve the BBM, PDEs.

Steps: First we consider the the BBM, PDEs of the type

$$a_1 w_t(z,t) + b_1 w_z(z,t) + c_1 w(z,t) w_z(z,t) - d_1 w_{zzt}(z,t) = g_1(z,t), \tag{12}$$

With initial condition,

$$w(z,0) = f_1(z), \quad 0 \le z \le 1, \tag{13}$$

and boundary conditions:

$$w(0,t) = h_0(t), \; w(1,t) = h_1(t), \quad for\ all\ t \ge 0, \tag{14}$$

where,

a_1, b_1, c_1, d_1 are real constant and $f_1(z), h_0(t), h_1(t), g_1(z,t)$ are real–valued continuous function on $[0,1) \times [0,1)$.

Let,

$$\mathrm{w_{zzt}(z,t) \approx \Psi^T(t) P \Psi(z)}, \tag{15}$$

where,

the value of $\Psi^T(t), \Psi(z), P$ described in the above equations (10), (8), (11) respectively.

Where, P represent $N \times N$ MLWs coefficient matrix to be found. Now integrate equation (15) with regard to t from 0 to t.

$$\mathrm{w_{zz}(z,t) = w_{zz}(z,0)} + \int_0^t \Psi^T(t) P \Psi(z) dt, \tag{16}$$

now, integrate equation (16) with regard to z from 0 to z.

$$\mathrm{w_z(z,t)} = \mathrm{w_z(0,t)} + \mathrm{w_z(z,0)} - \mathrm{w_z(0,0)} + \int_0^z \int_0^t \Psi^T(t) P \Psi(z)\, dt dz \,, \tag{17}$$

similarly, integrate equation (17) with regard to z from 0 to z.

$$w(z,t) = w(0,t) + \mathrm{z}\big(\mathrm{w_z(0,t)} - \mathrm{w_z(0,0)}\big) + w(z,0) - w(0,0) + \int_0^z \int_0^z \int_0^t \Psi^T(t) P \Psi(z)\, dt dz dz \,, \tag{18}$$

put $z = 1$, in equation (18) we get,

$$w(1,t) = w(0,t) + \big(\mathrm{w_z(0,t)} - \mathrm{w_z(0,0)}\big) + w(1,0) - w(0,0) + \int_0^1 \int_0^1 \int_0^t \Psi^T(t) P \Psi(1)\, dt dz dz \,, \tag{19}$$

On putting conditions from equations (13) and (14) into equation (19) we get,

$$\mathrm{w_z(0,t)} - \mathrm{w_z(0,0)} = h_1(t) - h_0(t) + f_1(0) - f_1(1) - \int_0^1 \int_0^1 \int_0^t \Psi^T(t) P \Psi(1)\, dt dz dz \,, \tag{20}$$

Using equations (17, 18) and (20) we get,

$$\mathrm{w_z(z,t)} = \mathrm{w_z(z,0)} + h_1(t) - h_0(t) + f_1(0) - f_1(1) - \int_0^1 \int_0^1 \int_0^t \Psi^T(t) P \Psi(1)\, dt dz dz$$

$$+ \int_0^z \int_0^t \Psi^T(t) P \Psi(z)\, dt dz \,, \tag{21}$$

and,

$$w(z,t) = w(0,t) + z\left(h_1(t) - h_0(t) + f_1(0) - f_1(1) - \int_0^1 \int_0^1 \int_0^t \Psi^T(t) P \Psi(1)\, dt dz dz\right)$$

$$+ w(z,0) - w(0,0) + \int_0^z \int_0^z \int_0^t \Psi^T(t) P \Psi(z)\, dt dz dz \,, \tag{22}$$

Differentiate equation (22) with regard to t, we get,

$$\mathrm{w_t(z,t)} = \mathrm{w_t(0,t)} + z\left({h_1}'(t) - {h_0}'(t) - \int_0^1 \int_0^1 \Psi^T(t) P \Psi(1)\, dz dz\right) + \int_0^z \int_0^z \Psi^T(t) P \Psi(z)\, dz dz \,, \tag{23}$$

Substituting the resulting values from equations (15), and (21-23) into equation (12) and applying following collocation points

$$z_i, t_j = \frac{2i-1}{2M}, \;\; i,j = 1,2,\dots,M. \tag{24}$$

Now we solve the obtained system by using Mathematica 7.0. , and we get MLWs coefficients $p_{i,j}$, $i,j = 1,2,\dots,2^{k-1}$ and then putting these coefficient in equation (22) we get the approximate solution of consider BBM PDEs. The absolute error calculated by $|Exact\ solution - Approximate\ solution|$.

4. Test example

Example 1. Consider the BBM, PDE of the form [9]

$$w_t(z,t) + w_z(z,t) - 2w_{zzt}(z,t) = 0 \,, \tag{25}$$

With initial conditions,

$w(z,0) = e^{-z}, \quad 0 \le z \le 1,$ (26)

and, boundary conditions

$w(0,t) = e^{-t},\ w(1,t) = e^{-1-t}, \quad for\ all\ t \ge 0,$ (27)

This test example has exact solution: e^{-z-t} .

By using conditions given in equations (28) and (29), we solve the equation (27) by using MLWSCM discussed in this article we get the following system of algebraic equations:

$$\left(-e^{-t_j} - e^{-z_i} + e^{-1-t_j} - e^{-t_j} + 1 - e^{-1}\right) + z_i\left(-e^{-1-t_j} + e^{-t_j}\right) - z_i \int_0^1 \int_0^1 \Psi^T(t_j) P\Psi(1) dz dz$$
$$+ \int_0^{z_i} \int_0^{z_i} \Psi^T(t_j) P\Psi(z_i) dz dz - 2\Psi^T(t_j) P\Psi(z_i) - \int_0^1 \int_0^1 \int_0^{t_j} \Psi^T(t_j) P\Psi(1) dt dz dz$$
$$+ \int_0^{z_i} \int_0^{t_j} \Psi^T(t_j) P\Psi(z_i) dt dz = 0. \quad (28)$$

We solve the above system described in equation (8), for $k = 1, M = 6$ by using Mathematica software 7.0, we get the unknown matrix $P_{6\times 6}$. Substitute this matrix in equation (22) we get MLWsCS based approximate solution and compared with some existed methods results with their absolute error is presented in table-1. Exact and approximate solutions graph is represented in figure 1 and 2 respectively.

Table 1.

z	FDM[8]	HWM[8]	MLWsCS	Exact	Absolute error in FDM[8]	Absolute errors in HWM[8]	Absolute errors in MLWsCS
1/16	0.82942254	0.82957675	0.82897223	0.82902911	3.93E-04	5.47E-04	5.684E-05
3/16	0.73277957	0.73244494	0.73144137	0.73161562	1.16E-03	8.29E-04	1.742E-04
5/16	0.64756878	0.64621771	0.64534316	0.64564852	1.92E-03	5.69E-04	3.053E-04
7/16	0.57245382	0.56995069	0.56938845	0.56978282	2.67E-03	1.67E-04	4.639E-04
9/16	0.50625604	0.50267060	0.50216799	0.50283157	3.42E-03	1.60E-04	6.635E-04
11/16	0.44793598	0.44342497	0.44283339	0.44374731	4.18E-03	3.22E-04	9.139E-04
13/16	0.39657711	0.39130681	0.39038839	0.39160562	4.97E-03	2.98E-04	1.217E-03
15/16	0.35137142	0.34546659	0.34402782	0.34559075	5.78E-03	1.44E-04	1.562E-03

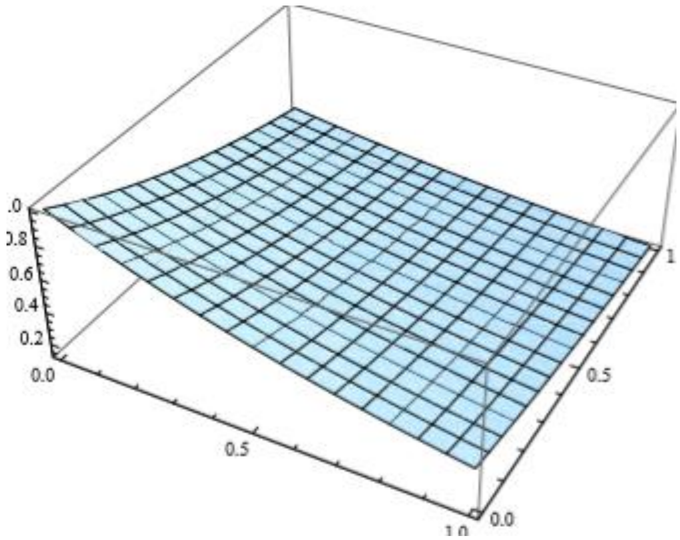

Figure 1: Graph of exact solution for test example 1

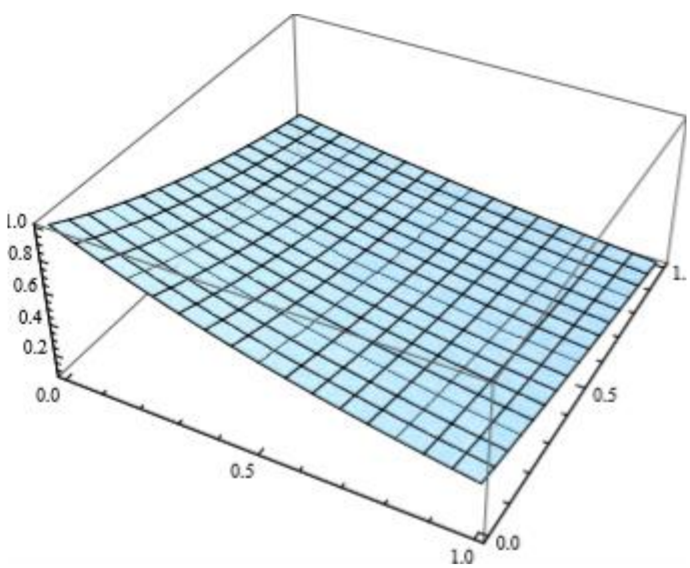

Figure 2: Graph of MLWsCS solution for test example 1

Conclusions

MLWsCS is described in this article for calculating the BBM, PDE for physical conditions, which is an achievement in new research. The scheme is evaluated for PDEs of the type BBM. If we increase no. of MLWs bases then we get further improvement in approximation solution. The proposed scheme is computationally effective and provides more improvements in numerical results of PDEs which is approved by the approximate solution for the given test example. The solution received from the MLWsCS becomes more reliable if we use more MLWs basis. In experimental test example, we use Mathematica 7.0 to perform the calculations, and the CPU running time approximately 7 to10 seconds, where used processor is Intel Core i3, 5th generation.

References

[1]. Lepik, U., (2011) 'Solving pdes with the aid of two-dimensional Haar wavelets', *Journal of Computers &Mathematics and with Applications*, Vol. 61, No. 7,pp. 1873–1879. Doi: 10.1016/j.camwa.2011.02.016.

[2]. Burden R. L, and Faires J.D. (1993). *Numerical analysis*, Boston, MA: PWS, USA.

[3]. Olver P. J. (1986) *Applications of Lie groups to differential equations*, Berlin: Springer.

[4]. Salkuyeh, D. K. (2006) 'On the finite difference approximation to the convection-diffusion equation', *Applied Mathematics and Computation*, Vol. 179, No. 1, pp. 79–86. Doi: 10.1016/j.amc.2005.11.078.

[5]. Rogers, C., Shadwick, W. F. (1982) *Bäcklund transformations and their application*, Academic Press, New York.

[6]. Adomian, G. (1988) 'A review of the decomposition method in applied mathematics', *Journal of mathematical analysis and applications*, Vol. 135 , No. 2, pp. 501-544.

[7]. Abbasbandy, S., and Shirzadi, A. (2010) 'The first integral method for modified Benjamin–Bona–Mahony equation', *Communications in Nonlinear Science and Numerical Simulation*, Vol. 15, No. 7, pp. 1759-1764.

[8]. Shiralashetti, S. C., Angadi, L. M., Deshi, A. B., and Kantli, M. H. (2016) 'Haar wavelet method for the numerical solution of Benjamin-Bona-Mahony equations', *Journal of Information and Computing Sciences*, Vol. 11, No. 2, pp. 136-145.

[9]. Shiralashetti, S. C., and Kumbinarasaiah, S. (2019) ' Laguerre wavelets collocation method for the numerical solution of the Benjamina–Bona–Mohany equations', *Journal of Taibah University for science*, Vol. 13, No. 1, pp. 9-15.

[10]. Kumar, A., & Verma, S. R. (2023). A Numerical Approach Based on Modified Lucas Wavelets for Functional Variational Problems Through Integral Operational Matrix. *International Journal of Applied and Computational Mathematics*, *9*(6), 138.

[11]. Kumar, A. (2023). The Operational Matrix Method Based on Modified Lucas Wavelets for the Approximation Solution of Initial and Boundary Value Problems. *International Journal of Applied and Computational Mathematics*, *9*(6), 147.

[12]. Kumar, A. (2022). A Computational Derivative Operational Matrix Technique for Solving Second-Order Lane-Emden Type Differential Equations via Modified Lucas Wavelets Basis. *Mathematical Statistician and Engineering Applications*, *71*(3), 821-835.

[13]. Singh, A. K., Madhulika, M., Arora, R. (2021). Numerical Solution of Integro-Differential Equations by using Euler's Operational Matrix of Integration. *Mathematics in Engineering, Science and Aerospace (MESA).* Vol.12, No. 2, pp. 1-11.

[14]. Arora, R. Madhulika, Singh, A. K. (2022). Numerical Method of Abel's Type Integral Equations with Euler's Operational Matrix of Integration. Palestine Journal of Mathematics. Vol. 11(Special Issue I), 13-23.