Uttaranchal Journal of Applied and Life Sciences
U. J. App. Life. Sci. Vol: 4 (1): 1-8 **Dec 2023**

An Overview of Mamadu-Njoseh Wavelets and its Properties for Numerical Computations

Ashish Rayal [1*] , Monika Anand [2] , Kunjana Chauhan [3] , Prinsa [4]

[1,2,3,4] Department of Mathematics, School of Applied and Life Sciences,
Uttaranchal University, Dehradun-248007, Uttarakhand, India.
***Corresponding Author:** ashishrayal@uumail.in

Received: (8 August 2023) Revised: (10 Oct 2023) Accepted: (1 Nov 2023)

Abstract: The primary purpose of this article is to introduce the Mamadu-Njoseh wavelets which make a series of orthogonal wavelets. The orthogonal wavelets have a huge contribution in numerical analysis as well as in approximation theory. We review and discuss several essential aspects of Mamadu-Njoseh wavelets such as the orthogonality, admissibility, and regularity condition with function approximation. Also, the operational matrix of integral and derivative for Mamadu-Njoseh wavelets is constructed which are very useful for the simulation of several models.

Keywords: Mamadu-Njoseh wavelets; Function approximations; Operational matrices.

INTRODUCTION

Wavelet theory is a growing area in the field of finance, applied science and engineering. Wavelets (Debnath, 2002; Datta and Mohan, 1995) are unique varieties of compactly supported oscillatory functions. It allows the exact representation of a variety of operators and functions. The ultimate speculate behind the exploration of wavelets bases is that the Fourier series cannot analyse the signals in time domain as well as in frequency domain. The noteworthy utility of the wavelets transform is its capability to analyze the signals in time and frequency domain simultaneously. Wavelets have shown to be very efficient and powerful for the explore of numerical simulations of various problems in disparate fields like engineering, economics, biology, physics and so on. Therefore, the wavelet-based approach is an efficient numerical technique for simulating variety of models and it requires a less amount of calculation. In recent years, several numerical approaches related to wavelets like Taylor wavelets (Rayal, 2023a), Muntz wavelets (Rayal and Verma, 2022a), Gegenbauer wavelets (Rayal and Verma, 2022b), Bernstein Wavelets (Rayal, 2023b), and Legendre wavelets (Rayal and Verma, 2020) have been employed to acquire the solutions of various applications of real life such as time-frequency analysis, remote sensing, signal analysis (Chui, 1997), ideal command, system analysis (Graps, 1995), data segmentation (Lucie et al., 1994), pattern recognition, and optimal control (Moradi et al., 2019). In applied phenomena such as signal processing, wavelets are more effective tool for approximating signals (Daubechies, 1990). More applications of wavelets are given in the references (Rayal and Verma, 2021; Rayal et al., 2022c; Rayal, 2023c).

In the recent sequence of wavelets, we will introduce the Mamadu-Njoseh wavelets (Iweobodo et al., 2023) with Mamadu-Njoseh polynomials and review its several essential characteristics such as orthogonality, regularity, and admissibility condition which are very beneficial for the numerical computations of several real-life phenomena.

MAMADU-NJOSEH WAVELETS

The Mamadu-Njoseh wavelets **(**MNWs) $\Im_{n,m}(\sigma)$ for $n = 1, 2, ..., 2^{k-1}$, positive integer k, and $m = 0, 1, ..., M-1$ is defined on [0, 1) as (Iweobodo et al., 2023)

$$\Im_{n,m}(\sigma) = \begin{cases} 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} MN_m(2^k \sigma + 1 - 2n), & \left[\frac{2n-2}{2^k}, \frac{2n}{2^k}\right), \\ 0, & \text{elsewhere} \end{cases} \tag{1}$$

where MN_m are the Mamadu-Njoseh polynomials (MNPs) (Mamadu and Njoseh, 2016) of degree *m* under the weighted function $w(\sigma) = \sigma^2 + 1$ on the interval [-1, 1]. Therefore, the first six MNPs (Njoseh and Mamadu, 2016) are given as

$$MN_0(\sigma) = 1$$
$$MN_1(\sigma) = \sigma$$
$$MN_2(\sigma) = \frac{1}{3}(5\sigma^2 - 2)$$
$$MN_3(\sigma) = \frac{1}{5}(14\sigma^3 - 9\sigma)$$
$$MN_4(\sigma) = \frac{1}{648}(3213\sigma^4 - 2898\sigma^2 + 333)$$
$$MN_5(\sigma) = \frac{1}{136}(1221\sigma^5 - 1410\sigma^3 + 325\sigma).$$

By using Equation (1), we can obtain the first five MNWs for M=5 and k=1 as

$$
\left.\begin{aligned}
\Im_{1,0}(\sigma) &= \frac{2}{\sqrt{\pi}} \\
\Im_{1,1}(\sigma) &= \frac{2}{\sqrt{\pi}}\left(2\sigma-1\right) \\
\Im_{1,2}(\sigma) &= \frac{2}{3\sqrt{\pi}}\left(20\sigma^{2}-20\sigma+3\right) \\
\Im_{1,3}(\sigma) &= \frac{2}{5\sqrt{\pi}}\left(122\sigma^{3}-168\sigma^{2}+66\sigma-5\right) \\
\Im_{1,4}(\sigma) &= \frac{2}{648\sqrt{\pi}}\left(51408\sigma^{4}-102816\sigma^{3}+65520\sigma^{2}-14112\sigma+648\right).
\end{aligned}\right\} \quad (2)
$$

Also, the above set of MNWs are depicted in the figure 1.

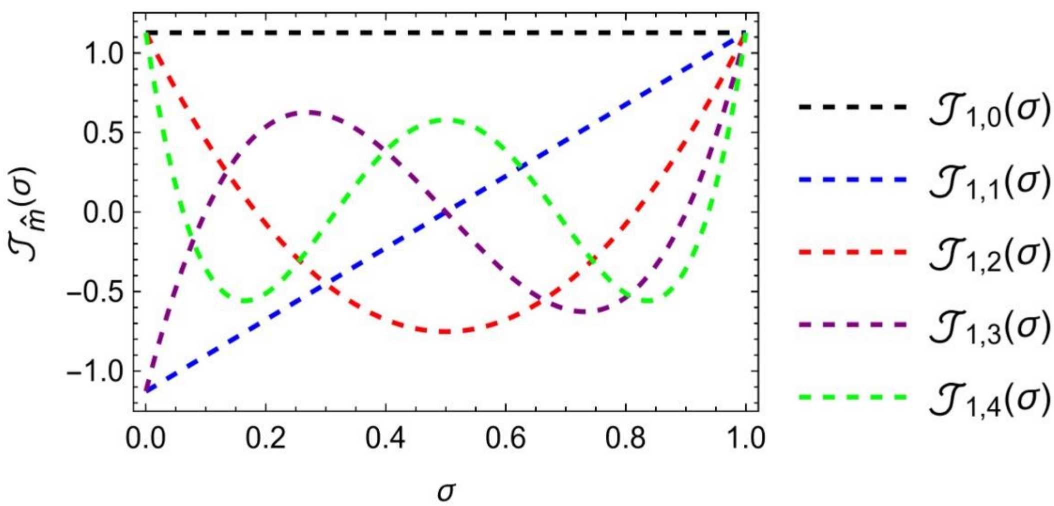

Figure 1: Graph of MNWs for k=1 and M=5.

- Some important characteristics of MNWs are reviewed and expressed using (Iweobodo et al., 2023) as:

a) Orthogonality Condition

The MNWs $\Im_{n,m}(\sigma)$ defined over a domain [0,1) under the weighted function $w_{n,m}(\sigma) = w(2^{k}\sigma - 2n + 1)$ satisfies the orthogonality condition as

$$\left\langle \Im_{n,m}(\sigma), \Im_{\bar{n},\bar{m}}(\sigma) \right\rangle_{w_{n,m}(\sigma)} = \int_0^1 \Im_{n,m}(\sigma)\Im_{\bar{n},\bar{m}}(\sigma)\, w_{n,m}(\sigma)d\sigma$$

$$= \begin{cases} 1, & n=\bar{n}, m=\bar{m} \\ 0, & otherwise \end{cases};$$

where $\langle .,. \rangle_{w_{n,m}(\sigma)}$ represents the inner product in $L^2[0,1]$.

b) Regularity Condition

The regularity condition requires that the wavelets should be locally concentrated and smooth in frequency as well as in time domain, i.e.,

$$\int_0^1 \Im_{n,m}(\sigma)\, d\sigma \to 0.$$

c) Admissibility Condition

The admissibility condition for the given wavelet is defined as

$$\int \frac{|\Im(\varpi)|^2}{|\varpi|} d\varpi < \infty,$$

where $\Im(\varpi)$ is the Fourier transform (FT) of the considered wavelets given by

$$\Im(\varpi) = \int_R \Im(\sigma)\, \mathrm{e}^{-j\varpi\sigma}\, d\sigma.$$

The FT of the considered wavelets at the origin is zero and it is given as

$$\Im(0) = \int_{-\infty}^{\infty} \Im(\sigma) d\sigma = 0.$$

FUNCTION APPROXIMATION VIA MNWs

Any arbitrary function $\zeta(\sigma)$ over [0, 1) is approximated via MNWs as

$$\zeta(\sigma) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \mu_{n,m}\, \Im_{n,m}(\sigma). \tag{3}$$

For numerical computation, the truncation version of Equation (3) is provided as

$$\zeta(\sigma) \cong \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} \mu_{n,m}\, \Im_{n,m}(\sigma) = U^T \Im_{\hat{m}}(\sigma),$$

where U & $\Im_{\hat{m}}(\sigma)$ are $(2^{k-1}M\times 1)$ order matrices provided as

$$U=\left[\mu_{1,0},\mu_{1,1},...,\mu_{1,M-1};\mu_{2,0},\mu_{2,1},...,\mu_{2,M-1};...;\mu_{2^{k-1},0},\mu_{2^{k-1},1},...,\mu_{2^{k-1},M-1}\right]^{T},$$

$$\Im_{\hat{m}}(\sigma)=\left[\Im_{1,0}(\sigma),\Im_{1,1}(\sigma),...,\Im_{1,\mathrm{M}-1}(\sigma);\Im_{2,0}(\sigma),\Im_{2,1}(\sigma),...,\Im_{2,M-1}(\sigma);...;\Im_{2^{k-1},0}(\sigma),\Im_{2^{k-1},1}(\sigma),...,\Im_{2^{k-1},\mathrm{M}-1}(\sigma)\right]^{T}.$$

Here, we take $\hat{\mathrm{m}}=2^{k-1}M$ as the total number of MNWs.

OPERATIONAL MATRIX OF INTEGRAL FOR MNWs

The integral operational matrix $\mathbf{P}_{\hat{\mathrm{m}}\times\hat{\mathrm{m}}}$ for MNWs is obtained by function approximation as

$$\int_0^{\sigma}\Im_{\hat{\mathrm{m}}\times 1}(\sigma)\,\mathrm{d}\sigma=\mathrm{P}_{\hat{\mathrm{m}}\times\hat{\mathrm{m}}}\Im_{\hat{\mathrm{m}}\times 1}(\sigma),$$

where $\Im_{\hat{\mathrm{m}}\times 1}(\sigma)$ is the MNWs and integral operational matrix $\mathbf{P}_{\hat{\mathrm{m}}\times\hat{\mathrm{m}}}$ is determined by

$$\mathrm{P}_{\hat{\mathrm{m}}\times\hat{\mathrm{m}}}=\left\langle \ell_{\hat{\mathrm{m}}\times 1}(\sigma),\Im_{1\times\hat{\mathrm{m}}}(\sigma)\right\rangle_{\mathrm{w}_{\mathrm{n,m}}(\sigma)},$$

where

$$\ell_{\hat{\mathrm{m}}\times 1}(\sigma)=\int_0^{\sigma}\Im_{\hat{\mathrm{m}}\times 1}(\sigma)\,\mathrm{d}\sigma.$$

- In order to demonstrate the computation procedure, we take $\hat{\mathrm{m}}=5$ $(\mathrm{k}=1,\mathrm{M}=5)$.

By using Equation (3), we have the following set of MNWs

$$\Im_{5\times 1}(\sigma)=\begin{bmatrix}\frac{2}{\sqrt{\pi}}\\ \frac{2}{\sqrt{\pi}}(2\sigma-1)\\ \frac{2}{3\sqrt{\pi}}\left(20\sigma^2-20\sigma+3\right)\\ \frac{2}{5\sqrt{\pi}}\left(122\sigma^3-168\sigma^2+66\sigma-5\right)\\ \frac{2}{648\sqrt{\pi}}\left(51408\sigma^4-102816\sigma^3+65520\sigma^2-14112\sigma+648\right)\end{bmatrix}. \tag{4}$$

Now, calculate the term $\ell_{5\times 1}(\sigma)$ as

$$\ell_{5\times 1}(\sigma)=\int_0^{\sigma}\Im_{5\times 1}(\sigma)d\sigma$$

$$=\begin{bmatrix} \frac{2\sigma}{\sqrt{\pi}} \\ \frac{2}{\sqrt{\pi}}\left(\sigma^2-\sigma\right) \\ \frac{2}{9\sqrt{\pi}}\left(20\sigma^3-30\sigma^2+9\sigma\right) \\ \frac{2}{5\sqrt{\pi}}\left(28\sigma^4-56\sigma^3+33\sigma^2-5\sigma\right) \\ \frac{2}{135\sqrt{\pi}}\left(2142\sigma^5-5355\sigma^4+4550\sigma^3-1470\sigma^2+135\sigma\right) \end{bmatrix}.$$

Then, the integral operational matrix $P_{5\times 5}$ is calculated as

$$P_{5\times 5}=\int_0^1\left(\ell_{5\times 1}(\sigma)\Im_{1\times 5}(\sigma)\right)w_{n,m}(\sigma)d\sigma$$

$$=\begin{bmatrix} \frac{8}{3\pi} & \frac{16}{15\pi} & 0 & 0 & 0 \\ -\frac{4}{5\pi} & 0 & \frac{68}{315\pi} & 0 & 0 \\ -\frac{8}{27\pi} & -\frac{104}{315\pi} & 0 & \frac{296}{2835\pi} & 0 \\ \frac{4}{75\pi} & 0 & -\frac{548}{4725\pi} & 0 & \frac{1312}{22275\pi} \\ \frac{16}{405\pi} & \frac{4}{135\pi} & 0 & -\frac{868}{13365\pi} & 0 \end{bmatrix}.$$

In such a way, we can obtain integral operational matrix of MNWs for any order.

OPERATIONAL MATRIX OF DERIVATIVE FOR MNWs

The derivative operational matrix $D_{\hat{m}\times\hat{m}}$ for MNWs is obtained by function approximation as

$$\frac{d}{d\sigma}\Im_{\hat{m}\times 1}(\sigma)=D_{\hat{m}\times\hat{m}}\Im_{\hat{m}\times 1}(\sigma),$$

where $\Im_{\hat{m}\times 1}(\sigma)$ is the MNWs and derivative operational matrix $D_{\hat{m}\times\hat{m}}$ is determined by

$$D_{\hat{m}\times\hat{m}}=\left\langle\kappa_{\hat{m}\times 1}(\sigma),\Im_{1\times\hat{m}}(\sigma)\right\rangle_{w_{n,m}(\sigma)},$$

where

$$\kappa_{\hat{m}\times 1}(\sigma)=\frac{d}{d\sigma}\Im_{\hat{m}\times 1}(\sigma).$$

- In order to demonstrate the computation procedure, we take $\hat{m}=5(k=1,M=5)$.

First, calculate the term $\kappa_{5\times1}(\sigma)$ by using Equation (4) as

$$\kappa_{5\times1}(\sigma)=\frac{d}{d\sigma}\Im_{5\times1}(\sigma)$$
$$=\begin{bmatrix} 0 \\ \frac{4}{\sqrt{\pi}} \\ \frac{2}{3\sqrt{\pi}}(40\sigma-20) \\ \frac{2}{5\sqrt{\pi}}(336\sigma^2-336\sigma+66) \\ \frac{2}{648\sqrt{\pi}}(205632\sigma^3-308448\sigma^2+131040\sigma-14112) \end{bmatrix}.$$

Then, the derivative operational matrix $D_{5\times5}$ is calculated as

$$D_{5\times5}=\int_0^1\left(\kappa_{5\times1}(\sigma)\Im_{1\times5}(\sigma)\right)w_{n,m}(\sigma)d\sigma$$
$$=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{32}{3\pi} & 0 & 0 & 0 & 0 \\ 0 & \frac{128}{9\pi} & 0 & 0 & 0 \\ \frac{416}{25\pi} & 0 & \frac{1088}{75\pi} & 0 & 0 \\ 0 & \frac{2192}{135\pi} & 0 & \frac{10064}{675\pi} & 0 \end{bmatrix}.$$

In such a way, we can obtain derivative operational matrix of MNWs for any order.

CONCLUSION

In this article, we introduced some rich properties related to MNWs with the concept of function approximation. The notion of function approximation allows to approximate the unknown functions through wavelet coefficients and known wavelets. These properties are very useful in computing higher order integrals and derivatives of MNWs. We can employ the above characteristics whenever required, some MNWs or its derivative with corresponding provided related functions. Since the operational matrices of integral and derivative for MNWs contain many zeroes, therefore these MNWs would provide an easy implementation to many numerical computations. These wavelets may be useful for approximating polynomial functions as well as for solving complex nonlinear problems.

REFERENCES

1. Chui C.K. (1997); Wavelets: a mathematical tool for signal analysis, Society for Industrial and Applied Mathematics.
2. Datta K.B., Mohan B.M. (1995); Orthogonal functions in systems and control, World Scientific, 9:127-155.
3. Daubechies I. (1990); The wavelet transforms, time-frequency localization, and signal analysis, IEEE transactions on information theory, 36(5):961-1005.
4. Debnath L. (2002); Wavelets Transform and Their Applications, Birkhauser, Boston.
5. Graps A. (1995); An introduction to wavelets, IEEE computational science and engineering, 2(2):50-61.
6. Iweobodo D. C., Njoseh I. N., Apanapudor J. S. (2023) A New Wavelet-based Galerkin method of weighted residual function for the numerical solution of one-dimensional differential equations, Mathematics and Statistics, 11(6):910-916.
7. Lucier B.J., Kallergi M., Qian W., DeVore R.A., Clark R.A., Saff E.B., Clarke L.P. (1994); Wavelet compression and segmentation of digital mammograms, Journal of digital imaging, 7(1):27-38.
8. Mamadu J.E., Njoseh I.N. (2016); Numerical solutions of Volterra equations using Galerkin method with certain orthogonal polynomials, Journal of Applied Mathematics and Physics, 4(2):367-382.
9. Moradi L., Mohammadi F., Baleanu D. (2019); A direct numerical solution of time-delay fractional optimal control problems by using Chelyshkov wavelets, Journal of Vibration and Control, 25(2):310-324.
10. Njoseh I.N., Mamadu E.J. (2016); Numerical solutions of fifth order boundary value problems using Mamadu-Njoseh polynomials, Siene World Journal, 11(4):21-24.
11. Rayal A. (2023a); An effective Taylor wavelets basis for the evaluation of numerical differentiations, Palestine Journal of Mathematics, 12:551-568.
12. Rayal A. (2023c); Muntz Wavelets Solution for the Polytropic Lane–Emden Differential Equation Involved with Conformable Type Fractional Derivative, International Journal of Applied and Computational Mathematics, 9(4):50.
13. Rayal A., Joshi B.P., Pandey M., Torres D.F. (2023b); Numerical Investigation of the Fractional Oscillation Equations under the Context of Variable Order Caputo Fractional Derivative via Fractional Order Bernstein Wavelets, Mathematics, 11(11):2503.
14. Rayal A., Tamta S., Rawat S., Kashif M. (2022c); Numerical view of Lucas-Lehmer polynomials with its characteristics, Uttaranchal Journal of Applied and Life Sciences, 3:66-75.
15. Rayal A., Verma S.R. (2020); Numerical analysis of pantograph differential equation of the stretched type associated with fractal-fractional derivatives via fractional order Legendre wavelets, Chaos, Solitons & Fractals, 139:110076.
16. Rayal A., Verma S.R. (2021); An approximate wavelets solution to the class of variational problems with fractional order, Journal of Applied Mathematics and Computing, 65(1-2):735-769.
17. Rayal A., Verma S.R. (2022a); Numerical study of variational problems of moving or fixed boundary conditions by Muntz wavelets, Journal of Vibration and Control, 28(1-2):214-229.
18. Rayal A., Verma S.R. (2022b); Two-dimensional Gegenbauer wavelets for the numerical solution of tempered fractional model of the nonlinear Klein-Gordon equation, Applied Numerical Mathematics, 174:191-220.